ASYMPTOTIC OF THE FLOW UPON SHOCK INCIDENCE ON A WEDGE CAVITY

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The asymptotic of the motion originating because of shock incidence on a wedge cavity in a metal is investigated as the wave amplitude tends to zero. It has been shown in [1] that the flow is hence divided into two domains. The principal term governing the flow in the first domain agrees with the acoustic approximation. The flow in the second domain is described by incompressible fluid equations in the principal term. Determination of the flow in the second domain is reduced herein to the solution of a singular nonlinear integral equation. A numerical solution is found for a series of values of the cavity aperture.

1. A shock, parallel to the xz plane and with constant pressure $p = p_1$, proceeds over a metal medium with a cutout wedge cavity whose edge coincides with the z axis, the plane of symmetry with the yz plane, and whose aperture is 2γ ($\gamma < \pi/2$). The equation of state of the medium is

$$p = \frac{\rho_0 c_0^2}{\varkappa}(\delta^\varkappa S - 1) \tag{1.1}$$

The velocity ahead of the wave is $\mathbf{u} = 0$, the pressure is $p = 0$, the density is $\rho = \rho_0$, the speed of sound is $c = c_0$, the relative density is $\delta = 1$, and the entropy quantity is $S = 1$. At the time $t = 0$, the shock front intersects the edge of the cavity. For $t > 0$ the flow in the metal becomes two-dimensional and self-similar, i.e., the gasdynamics functions depend on two variables $\xi = x/t$, $\eta = y/t$. Since a plane shock of constant amplitude moves over a substance at rest, then the flow behind the shock front will also have potential. By virtue of the self-similarity, let us represent the flow potential as

$$\varphi(x, y, t) = t\Phi(\xi, \eta) \tag{1.2}$$

Determining the flow originating behind the shock front is possible by numerical integration of the appropriate system of partial differential equations with two independent variables. However, under an essential assumption about the smallness of the ratio

$$\varepsilon = p_1 / \rho_0 c_0^2 \ll 1 \tag{1.3}$$

the asymptotic of the flow as $\varepsilon \to 0$ can be obtained. This problem was considered in [1] from this viewpoint. Starting from the smallness of the value of ε, the acoustic approximation was written down. It turned out that it could be an approximation of the flow only in a domain I, where

$$\varepsilon r^{\alpha-2} \ll 1, \quad r = \sqrt{\xi^2 + \eta^2} / c_0, \quad \alpha = \pi / 2(\pi - \gamma) \quad (1/2 < \alpha < 1) \tag{1.4}$$

The asymptotic of the flow in this domain as $r \to 0$ appears thus:

$$\Phi \approx 2^{2-\alpha}\varepsilon c_0^2 \cos \alpha\gamma \cos \alpha(\pi - \theta) r^\alpha / (\alpha - 1)\pi$$

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$$\delta - 1 \approx 4\varepsilon \cos \alpha\gamma \cos \alpha (\pi - \theta) r^{\alpha} / (\alpha - 1) \pi \qquad (1.5)$$
$$\theta = \text{arc tg } \xi / \eta.$$

In order to continue the solution into the domain II, where

$$\varepsilon r^{\alpha - 2} \approx O(1) \qquad (1.6)$$

a change of variable was performed which took account of the definition (1.6) of the domain II as well as the asymptotic (1.5)

$$\Phi = \varepsilon^{2/(2-\alpha)} \Phi_1, \quad \xi = \varepsilon^{1/(2-\alpha)} \xi_1, \quad \eta = \varepsilon^{1/(2-\alpha)} \eta_1, \quad r = \varepsilon^{1/(2-\alpha)} R, \qquad (1.7)$$
$$- 1 = \varepsilon^{2/(2-\alpha)} \Delta_1$$

A domain of finite values of R, hence, corresponds to the domain II, and the passage into domain I corresponds to the passage to the limit $R \to \infty$. As follows from substituting (1.7) into the system of gas-dynamics equations, the function $\Phi_1(\xi_1, \eta_1)$ should satisfy the Laplace equation in the principal term as $\varepsilon \to 0$, and for finite values of R, i.e., the flow in this domain is described in the principal term by incompressible fluid equations. The determination of the asymptotic of the flow in this domain as $\varepsilon \to 0$ is the purpose of the present paper.

Because of the flow symmetry, it is sufficient to examine a domain bounded by the axis of symmetry and an unknown free boundary $\eta_1 = \eta_1(\xi_1)$, $\xi_1 \geq 0$, after which the problem can be formulated thus: for a given angle γ, find the equation of the free boundary $\eta_1 = \eta_1(\xi_1)$ and the harmonic function $\Phi_1(\xi_1, \eta_1)$, satisfying the boundary conditions on the free boundary

$$\frac{d\eta_1}{d\xi_1} = \frac{\Phi_{1\eta_1} - \eta_1}{\Phi_{1\xi_1} - \xi_1} \qquad (1.8)$$

$$p(\xi_1, \eta_1(\xi_1)) = 0 \qquad (1.9)$$

and on the axis of symmetry

$$\Phi_{1\xi_1} = 0 \qquad (1.10)$$

Moreover, for "merger" with the acoustic approximation as $R \to \infty$, the asymptotic

$$\Phi_1 \approx 2^{2-\alpha} c_0^2 \cos \alpha\gamma \cos \alpha (\pi - \theta) R^{\alpha} / (\alpha - 1) \pi \qquad (1.11)$$

should hold, and on the free boundary

$$\xi_1 \approx \text{tg } \gamma \eta_1 \qquad (1.12)$$

The subscript 1 is henceforth omitted.

2. The problem just formulated becomes similar to the problem of uniform submersion of a wedge in a half-space occupied by an ideal incompressible fluid by a substitution. Z. N. Dobrovol'skaya solved the problem about a wedge by reducing it to seeking the solution of a certain singular nonlinear integral equation [2, 3]. This method is also used herein.

The complex potential

$$F(\zeta) = \Phi(\xi, \eta) + i\Psi(\xi, \eta) \qquad (2.1)$$

is considered in the plane of the complex variable $\zeta = \xi + i\eta$.

The flow domain in the ζ plane is represented as the image for the conformal mapping $\zeta = \zeta(w)$ of the upper half-plane of the complex variable $w = u + iv$. This mapping is constructed so that the origin ($u = 0$, $v = 0$) goes into the vertex of the free boundary ζ_A ($\xi = 0$, $\eta = \eta_A$), the real positive half-axis ($u \geq 0$, $v = 0$) goes into the axis of flow symmetry ($\xi = 0$, $\eta \leq \eta_A$), the negative half-axis ($u \leq 0$, $v = 0$) goes into the free boundary, and the point $w = \infty$ into the point $\zeta = \infty$. Taking account of the asymptotic (1.12), it hence follows that

$$\arg \zeta'(u) = -\pi / 2 \quad \text{for} \quad 0 \leqslant u < \infty \qquad (2.2)$$

$$\arg \zeta'(u) \to -(\pi/2+\gamma) \quad \text{for} \quad u \to -\infty \tag{2.3}$$

(here, and henceforth, the prime ' denotes the derivative).

The potential $F(\zeta)$ hence, goes over into the function $F(w)$ defined in the upper w half-plane. Relations connecting the functions $\zeta(w)$ and $F(w)$

$$\operatorname{Re}[iF'(u)] = \overline{\operatorname{Re}}[i\zeta'(u)\bar{\zeta}(u)] \quad \text{for} \quad -\infty < u < 0 \tag{2.4}$$

$$p = 0 \qquad \text{for} \quad -\infty < u < 0 \tag{2.5}$$

follow from the boundary conditions (1.8) and (1.9),

$$\operatorname{Re}[iF'(u)] = 0 \quad \text{for} \quad 0 < u < \infty \tag{2.6}$$

follows from the flow symmetry condition (1.10), and finally, the asymptotic

$$F(\zeta) = (2c_0)^{2-\alpha} \cos\alpha\gamma\,(i\zeta)^\alpha / (\alpha-1)\pi \quad \text{for} \quad \zeta \to \infty \tag{2.7}$$

following from (1.11) should be satisfied.

Therefore, the problem can be formulated thus: determine two analytic functions $\zeta(w)$ and $F(w)$ by means of the boundary conditions given on the real axis of the w plane and by the asymptotic at infinity. The determination of these functions allows one degree of arbitrariness: the variable w can be multiplied by an arbitrary positive constant.

The potential $F(w)$ can be eliminated from the boundary conditions, after which, a boundary-value problem is obtained for the single function $\zeta(w)$. The Wagner function [4]

$$h(\zeta) = \int_{\zeta_A}^{\zeta} \sqrt{d^2F/d\zeta^2}\, d\zeta \tag{2.8}$$

is used for this purpose.

The angle $\pi/4$ in the h plane will be the image of the flow domain in the ζ plane, where the free boundary is mapped onto the ray $0 \le \arg h(\zeta) \le \pi/4$, and the axis of symmetry is mapped onto the ray $\arg h(\zeta) = 0$ (see Appendix 1). Hence

$$h[\zeta(w)] = D_0 w^{1/4}, \quad D_0 > 0 \tag{2.9}$$

Therefore

$$\frac{d^2F}{d\zeta^2} = \left(\frac{dh}{d\zeta}\right)^2 = \left(\frac{dh}{dw}\right)^2 \left(\frac{d\zeta}{dw}\right)^{-2} = \frac{1}{16} D_0^2 w^{-3/2} \left(\frac{d\zeta}{d\omega}\right)^{-2} \tag{2.10}$$

and, since the velocity of the apex of the free boundary ζ_A coincides with the vector ζ_A, because of self-similarity, then

$$\frac{dF}{d\zeta} = \bar{\zeta}_A + \frac{D_0^2}{16} \int_0^w w^{-3/2} \left(\frac{d\zeta}{dw}\right)^{-1} dw \tag{2.11}$$

($\bar{\zeta}_A$ is the complex conjugate of ζ_A).

Now $F'(u)$ is eliminated from the boundary condition (2.4) by using the quadrature (2.11) and, consequently, an integral equation for just the single function $\zeta'(u)$ is obtained on the negative part of the real axis:

$$\operatorname{Re}\left\{i\zeta'(u) \int_0^u \{\overline{\zeta'(u)} - - D_0^2 / [16 u^{3/2} \zeta'(u)]\}\, du\right\} = 0 \tag{2.12}$$

where $\zeta'(u)$ is understood to be the limit value of $\zeta'(w)$ as $w \to u$. It is convenient to replace the complex function $\zeta'(u)$ by the real function $f(u)$ defined by the equality

$$f(u) = \arg \zeta'(u) + \pi/2 + \gamma \tag{2.13}$$

To do this, we introduce the function

$$q(w) = -i \ln [i w^{\gamma/\pi} \zeta'(w)] \tag{2.14}$$

It follows from (2.13) and (2.2) that

$$\operatorname{Re} q(w) = f(u) \text{ for } -\infty < u \leqslant 0, \operatorname{Re} q(w) = 0 \text{ for } 0 < u < \infty \tag{2.15}$$

By using the Schwartz integral

$$J(w) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{f(u)\, du}{u - w} \tag{2.16}$$

the function q(w) is reproduced in the whole upper w half-plane, and then, $\zeta'(w)$ is

$$\zeta'(w) = -iDw^{-\gamma/\pi} \exp [J(w)], \quad D > 0 \tag{2.17}$$

after which an integral equation for the function $f(u)$

$$\frac{df(u)}{du} = \frac{D_0^2 (-u)^{\gamma/\pi - 3/2} \exp[-J(u)]}{16D^2 \int_0^u [-u_1]^{-\gamma/\pi} \exp[J(u_1)]\, du_1} \tag{2.18}$$

is obtained as a result of passing to the limit as $w \to u$ in (2.17) according to the Sokhotskii theorem, and substituting the limit value into (2.12).

The singular integral J is understood in the principal value sense. There follows from the definition (2.13) and the asymptotic (2.5) that

$$f(-\infty) = 0 \tag{2.19}$$

After the function $f(u)$ has been found from (2.17) to the accuracy of the constant ζ_A, the mapping function $\zeta(w)$, the equation of the free boundary

$$\zeta(u) = \zeta_A - iD \int_0^u u_1^{-\gamma/\pi} \exp[J(u_1) + if(u_1)]\, du_1 \tag{2.20}$$

and the velocity distribution

$$F'(w) = \bar{\zeta}_A + i \frac{D_0^2}{16D} \int_0^w w_1^{\gamma/\pi - 3/2} \exp[-J(w_1)]\, dw_1 \tag{2.21}$$

are reproduced.

The pressure p is determined from the Bernoulli law.

The desired function $f(u)$ should be bounded because of the definition (2.13). Moreover, it is henceforth assumed that $f(u)$ tends to zero as $(-u)^{-k}$ as $u \to -\infty$, and its derivative $f'(u)$ as $(-u)^{-(k+1)}$, where $k > 0$. Then, the integral J is finite for each value of w, and tends to zero as $w \to -\infty$. Hence, the function $f(u)$ has a finite negative derivative for any value $u \neq 0$, and the asymptotics

$$f(u) \approx \frac{D_0^2}{8D^2} \frac{\pi - \gamma}{4\gamma - 3\pi} |u|^{(4\gamma - 3\pi)/2\pi}, \quad f'(u) \approx -\frac{D_0^2}{16D^2} \frac{\pi - \gamma}{\pi} |u|^{(4\gamma - 5\pi)/2\pi} \tag{2.22}$$

are valid as $u \to -\infty$.

Therefore, the assumptions made do not contradict (2.18) and $k = (4\gamma - 3\pi)/2\pi$. Since the derivative $f'(u)$ is negative for any value of u, then the function $f(u)$ increases monotonely from $f(0)$ to 0 as u varies between 0 and $-\infty$; hence, it turns out that the value of $f(0)$ should satisfy the inequality

$$\gamma - \pi < f(0) < \gamma - {}^3/_4\, \pi \tag{2.23}$$

Otherwise, it will follow from the asymptotic J(w) as $w \to 0$ that either $f(0) \to \infty$ or $f(u) \equiv 0$, which is inadmissible. The inequality (2.23) has a geometric meaning because the angle β between the axis of symmetry and the free boundary at its vertex is defined by the equality

$$\beta = \gamma - f(0) \tag{2.24}$$

To complete the formulation of the problem, it is necessary to find the value of the three, as yet undetermined, parameters (D_0, D, ζ_A). Their values are determined by the given asymptotic at infinity. Since $F'(\zeta) \to 0$ as $\zeta \to \infty$, then, it follows from (2.2)

$$\zeta_A = \frac{D_0^2}{16D} i \int_0^\infty u_1^{\gamma/\pi - 3/2} \exp[-J(u_1)]\, du_1 \tag{2.25}$$

According to (2.17) and (2.21), as $w \to \infty$, the quantity $\zeta(w) \to \infty$, and

$$F'(\zeta) \approx \frac{D_0^2}{D^\alpha} i^\alpha \frac{\pi^{2-\alpha}(\pi-\gamma)^{\alpha-1}}{8(2\gamma-\pi)} \zeta^{\alpha-1} \tag{2.26}$$

There follows from a comparison with the given asymptotic (2.7):

$$D_0^2 / D^\alpha = 16(\alpha)^{\alpha-1} c_0^{2-\alpha} \cos\alpha\gamma / \pi \tag{2.27}$$

It turns out that conditions (2.25) and (2.27) determine the flow completely, since upon compliance with these conditions the graphs of the free boundary and the velocity distribution in the $\xi\eta$ plane are independent of specific values of D_0 and D.

Indeed, let (D_{01}, D_1) and (D_0, D) be two pairs of numbers satisfying condition (2.27). Then

$$D_0 / D_{01} = (D / D_1)^{\alpha/2} = k^{-1/4} \tag{2.28}$$

where k is some positive number.

If $f(u)$ corresponds to (D_{01}, D_1), then $f(u/k)$ corresponds to (D_0, D). Hence, $\zeta(u)$ and $F'(\zeta)$, corresponding to (D_0, D) are defined by the formulas

$$\zeta(u) = \zeta_A - iD_1 \int_0^{u/k} u_1^{-\gamma/\pi} \exp[J(u_1) + if(u_1)]\, du_1 \tag{2.29}$$

$$F'(\zeta) = \bar{\zeta}_A + i\frac{D_{01}^2}{16D_1} \int_0^{w/k} w_1^{\gamma/\pi - 3/2} \exp[-J(w_1)]\, dw_1 \tag{2.30}$$

i.e., the shape of the free boundary and the velocity distribution did not vary in the ζ plane. Hence, $D_0^2/116\, D^2$ can be taken equal to one. But then,

$$D = c_0 [(\alpha)^{\alpha-1} \cos\alpha\gamma]^{1/(2-\alpha)} \pi^{1/(\alpha-2)}, \quad D_0 = 4D \tag{2.31}$$

3. Thus, the problem has been reduced to seeking the solution $f(u)$ of the integral equation (2.18) with the initial value (2.19). The function $f(u)$ should be a bounded, differentiable, and monotonely decreasing function of u

$$f(0) \leqslant f(u) < 0 \tag{3.1}$$

where

$$\gamma - \pi < f(0) < \gamma - {}^3/_4\, \pi \tag{3.2}$$

The numerical solution of (2.18) was carried out by an iteration method for a series of values of γ in the range $0 < \gamma < \pi/2$. Taking into account that the value obtained for $f(0)$ in some iteration step can emerge outside the admissible interval (3.2), the iteration process was constructed as follows. As a result of substituting the function $f_{n-1}(u)$ in the right side of (2.18) in the n-th step, let the function $f_{n1}(u)$ be obtained, and let

$$x = f_{n1}(0) / f_{n-1}(0) \tag{3.3}$$

Then, the function

$$f_n(u) = \frac{f_n(0)(1+u^2) f_{n1}(u)}{f_n(0) + f_n(0) u^2} \tag{3.4}$$

is obtained as the result of the n-th iteration, where

$$f_n(0) = f_{n-1}(0) - \frac{(\gamma - \pi - f_{n-1}(0))(\gamma - {}^3/_4\pi - f_{n-1}(0))(x-1)}{(\gamma - \pi - f_{n-1}(0)) - (\gamma - {}^3/_4\pi - f_{n-1}(0))x} \tag{3.5}$$

If $f_{n-1}(u)$ is a bounded function of u in the interval $-\infty < u < 0$, which satisfies condition (3.2) and the asymptotic (2.22), together with its derivative, then

$$J_{n-1}(u) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{f_{n-1}(u_1)}{u_1 - u} du_1 \to 0 \tag{3.6}$$

as $u \to -\infty$. Hence, as $u \to -\infty$ the functions $f_{n-1}(u)$ and $f_n(u)$ together with their derivatives will satisfy the asymptotic (2.22) by virtue of (2.18) and (3.5).

4. The function $f_{n1}(u)$ is determined by the equation

$$\frac{df_{n1}}{du} = \frac{(-u)^{(2\gamma - 3\pi)/2\pi} \exp[-J_{n-1}(u)]}{\int_0^u (-v)^{-\gamma/\pi} \exp[J_{n-1}(v)]\, dv} \qquad (f_{n1}(-\infty) = 0) \tag{4.1}$$

Since $f_{n-1}(0) = \gamma - \beta$, where β is some quantity satisfying the inequality $\frac{3}{4}\pi < \beta < \pi$, then as $u \to 0$

$$J_{u-1}(u) \approx [(\gamma - \beta)/\pi] \ln(-u), \quad \frac{df_{n1}}{du} \approx A(-u)^{2\beta/\pi - 5/2} \to \infty \tag{4.2}$$

as $u \to -\infty$

$$df_{n1}/du \sim (-u)^{(4\gamma - 5\pi)/2\pi} \tag{4.3}$$

For convenience in the numerical integration it is desirable to replace (4.1) by an equivalent equation or system of equations, where the interval of integration would be finite, and there would be no divergent integrals and infinite derivatives. This program was realized as follows.

The finite interval of integration $0 > u > -\infty$ was reduced to the finite interval $0 < \tau < 1$ by using the change of variable

$$u = -\tau^{\delta}(1-\tau)^{-\lambda} \qquad (\delta = 2\pi/(4\beta - 3\pi), \quad \lambda = \pi/(\beta - \gamma)) \tag{4.4}$$

In order to avoid divergent integrals, the Cauchy-type integral in the right side was represented as follows:

$$\frac{1}{\pi}\int_{-\infty}^{0} \frac{f_{n-1}(v)\, dv}{v-u} = \frac{1}{\pi}\int_{-\infty}^{0} \frac{1}{(v-u)}\left[f_{n-1}(v) - \frac{f_{n-1}(0)}{1-v}\right] dv + \frac{f_{n-1}(0)\ln(-u)}{\pi(1-u)} \tag{4.5}$$

The integral in the denominator of (4.1) was computed by integrating a differential equation for some function $\chi_n(u)$ related to the integral as follows:

$$\int_0^u (-v)^{-\gamma/\pi} \exp[J_{n-1}(v)]\, dv = (-u)^{(\pi-\beta)/\pi} / \chi_n(u) \tag{4.6}$$

Such a representation is connected with the fact that as $u \to 0$ the integral on the left tends to zero as $(-u)^{(\pi-\beta)/\pi}$, and to infinity as $u \to -\infty$.

TABLE 1

ν	β	η_A, km/sec
$11\pi/24$	2.8762	1.6563
$5\pi/12$	2.9148	1.9443
$\pi/3$	2.9915	2.2988
$\pi/4$	3.0468	3.5548
$\pi/6$	3.0905	4.8422
$\pi/12$	3.1260	7.5936
$\pi/24$	3.1384	11.6282

For smoothness of the computation as $\tau \to 1$, the function f_{n1} was replaced by the function R_{n1} in conformity with the formula

$$f = R\,[1 + (-u)^{(4\beta-3\pi)/2\pi}] \tag{4.7}$$

After performing the manipulations (4.4)-(4.7), Eq. (4.1) was replaced by the equivalent system of equations

$$\frac{d\chi_n}{d\tau} = \left[\frac{\delta}{\tau} + \lambda\frac{1}{1-\tau}\right]\left[\chi_n{}^2 L_{n-1}(\tau) + \frac{\pi-\beta}{\pi}\chi_n\right] \tag{4.8}$$

$$\frac{dR_n}{d\tau} = -\left[\delta + \lambda\frac{\tau}{1-\tau}\right]\left[\frac{\chi_n}{L_{n-1}(\tau)} + \frac{R_{n1}}{\delta}\right]\frac{1}{\tau + (1-\tau)^{\lambda/\delta}} \tag{4.9}$$

where

$$L_{n-1}(\tau) = (-u)^{u/[\lambda(u-1)]}\exp\left\{-\frac{1}{\pi}\int_0^1 \frac{[\delta(1-y)+\lambda y]\,y^{\delta-1}(1-y)^{-(\lambda+1)}}{[y^\delta(1-y)^{-\lambda} - \tau^\delta(1-\tau)^{-\lambda}]}\varphi_{n-1}(y)\,dy\right\}$$

$$\varphi_{n-1}(y) = \left[R_{n-1}(y)(1 + y(1-y)^{-\lambda/\delta}) - \frac{R_{n-1}(0)}{1 + y^\delta(1-y)^{-\lambda}}\right] \tag{4.10}$$

The integral in $L_{n-1}(\tau)$ is understood in the principal value sense, i.e.,

$$\int_0^1 \frac{[\delta(1-y)+\lambda y]\,y^{\delta-1}(1-y)^{-(\lambda+1)}}{[y^\delta(1-y)^{-\lambda} - \tau^\delta(1-\tau)^{-\lambda}]}\varphi(y)\,dy = \tag{4.11}$$

$$= \int_0^1 \left\{\frac{[\delta(1-y)+\lambda y]\,y^{\delta-1}(1-y)^{-(\lambda+1)}\varphi(y)\;y-\tau}{y^\delta(1-y)^{-\lambda} - \tau^\delta(1-\tau)^{-\lambda}} - \varphi(\tau)\right\}\frac{dy}{y-\tau} + \varphi(\tau)\ln\frac{1-\tau}{\tau}$$

Integration of the system (4.8), (4.9) was carried out in two steps.

1. Equation (4.8) was integrated between $\tau = 0$ and $\tau = 1$

$$\chi_n(0) = -\pi^{-1}(\pi-\beta)L_{n-1}(0), \quad \chi(1) = 0 \tag{4.12}$$

The point ($\tau = 0$, $\chi = \chi(0)$) is a saddle point, and the point ($\tau = 1$, $\chi = 0$) is a node for (4.1). Emergence from the point ($\tau = 0$, $\chi = \chi(0)$) is accomplished according to the asymptotic

$$\chi(\tau) = \chi(0) - \frac{(\pi-\beta)^3\delta\,\mathrm{ctg}\,(1-1/\delta)\pi}{\pi^2(2\beta-\pi)L_{n-1}^2(0)}\tau \tag{4.13}$$

the integral curve hence arrives at the point $\tau = 1$, $\chi = 0$ in conformity with the asymptotic

$$\chi_n(\tau) \approx \frac{\gamma-\pi}{\pi}\left[1 + \frac{\delta}{\lambda}(1-\tau) + C(1-\tau)^{(\pi-\gamma)/(\beta-\gamma)}\right](1-\tau) + \ldots \tag{4.14}$$

where C is a constant obtained as a result of the integration.

2. After the function $\chi_n(\tau)$ has been obtained, (4.9) was integrated between $\tau = 1$ and $\tau = 0$. As $\tau \to 1$, the function $R_{n1}(\tau) \to 0$. For a better computation of the asymptotic (4.14) it is expedient to represent the function $R_{n1}(\tau)$ as follows:

$$R_n(\tau) \approx \frac{2(\pi-\gamma)}{4\gamma-3\pi}(1-\tau)^2\left\{1 - (1-\tau)^{\lambda/\delta} + \left[1 - 4\frac{(\lambda-2\delta)(4\gamma-3\pi)}{(\lambda-3\delta)(4\beta-3\pi)} - \frac{\lambda\delta[\pi\chi_n(\tau) - (\gamma-\pi)(1-\tau)]}{\pi(\lambda-3\delta)(1-\tau)^2}\right](1-\tau) + \ldots\right\} \tag{4.15}$$

The value of the function $R_n(\tau)$ at $\tau = 0$ can be obtained only as a result of integration.

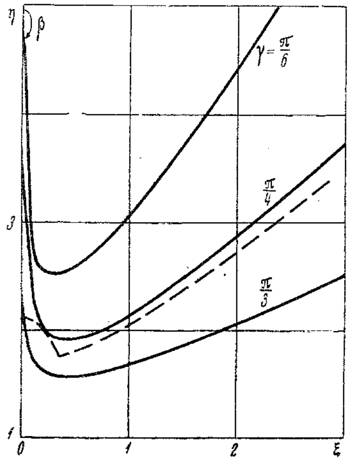

Fig. 1

Equations (4.8) and (4.9) were integrated by the Euler method with a conversion with a constant step h, and the integrals were computed by the Gauss formula with n nodes. It turns out that $h = 0.01$ and $n = 16$ assure sufficient accuracy.

5. In conformity with (4.4) and (4.7), the desired function became $R(\tau)$. The iteration formulas (3.3)–(3.5) were transformed correspondingly. A monotonely increasing function of τ, which satisfies the conditions

$$\gamma - \pi < R_0(0) < \gamma - {}^3/_4\pi, \quad R_0(\tau) \approx \frac{2(\pi-\gamma)}{4\gamma - 3\pi}(1-\tau)^2 \quad \text{for} \quad \tau \to 1 \qquad (5.1)$$

was taken as $R_0(\tau)$.

The iterations were carried out until the inequality

$$M(n) = \max | R_{n1}(\tau) / R_{n-1}(\tau) - 1 | \leqslant 0.001 \qquad (5.2)$$

was satisfied.

For the selected iteration method the function M(n) turned out to be monotonely decreasing and tending rapidly to zero. As the angle γ decreased the number of iterations needed to satisfy (5.2) grew.

The function $R(\tau)$ obtained was used to determine the profile of the free boundary in conformity with the equations

$$\begin{gathered}
\xi = \tau^{\delta(\pi-\beta)/\pi}(1-\tau)^{-\lambda(\pi-\gamma)/\pi}\xi_1, \quad \eta = \eta_A + \tau^{\delta(\pi-\beta)/\pi}(1-\tau)^{-\lambda(\pi-\gamma)/\pi}\eta_1 \\
\frac{d\xi_1}{d\tau} = a(\tau)\sin b(\tau) + \xi_1 c(\tau), \quad \frac{d\eta_1}{d\tau} = -a(\tau)\cos b(\tau) + \eta_1 c(\tau) \\
a(\tau) = D[\delta(\tau-1)/\tau - \lambda]L(\tau) \\
b(\tau) = R(\tau)[1 + \tau(1-\tau)^{-\lambda/\delta}] - \gamma \\
c(\tau) = (\beta-\pi)\delta/\pi\tau - (\pi-\gamma)\lambda/\pi(1-\tau)
\end{gathered} \qquad (5.3)$$

The value of η_A agreeing with the velocity at the vertex of the free boundary is determined according to (2.25) by the formula

$$\begin{gathered}
\eta_A = D\int_0^1 u^{(1/l+u/k)/(1-u)} \exp\left[\frac{1}{\pi}\int_0^1 \frac{\varphi(\tau_1)u_1}{u_1+u}\left(\frac{\delta}{\tau_1} + \frac{\lambda}{1-\tau_1}\right)d\tau_1\right]\left(\frac{l}{\tau} + \frac{k}{1-\tau}\right)d\tau \\
u = \tau^l(1-\tau)^{-k}, \quad u_1 = \tau_1^{\delta}(1-\tau_1)^{-\lambda}, \quad l = 2\pi/(2\beta-\pi), \quad k = \pi/(\pi-2\gamma)
\end{gathered} \qquad (5.4)$$

Profiles of the free boundaries were determined for a series of values of the angle γ. A table of the appropriate values of the angles γ, β, and the quantity η_A is presented above. Presented in the sketch are graphs of the free boundaries in the $\xi\eta$ plane defined in Section 1. The values of η_A and the graphs of the free boundaries correspond to the values $\varepsilon = 0.1$ and $c_0 = 5.5$ km/sec.

As follows from the table presented, as $\gamma \to \pi/2$, the angle β is close to 0.9π. On the other hand, for a cavity aperture $2\gamma = \pi$ the value of the angle is evidently $\beta = \pi/2$. Therefore, an arbitrarily small deviation of the angle γ from $\pi/2$ will result in a finite change in the angle β.

The numerical solution of this problem was performed on the BÉSM-4 electronic computer.

A computation of shock incidence on a wedge cavity was performed by a difference method for the value of $\gamma = \pi/4$ of the angle. The method developed by S. K. Godunov, A. V. Zabrodin, L. A. Pliner, and G. P. Prokopov to compute two-dimensional nonstationary gasdynamics problems in domains with complex geometry was used. The graph thus obtained for the free boundary is superposed on the figure by dashes for comparison. The author is grateful to A. V. Zabrodin, L. A. Pliner, and G. P. Prokopov for useful discussion of the results, and to N. V. Banichuk for programming and analysis.

APPENDIX

The angle

$$0 \leqslant \arg h(\zeta) < \pi/4$$

is the image of the flow domain in the h plane.

Indeed, it follows from (1.8) and (1.9) that the relationship

$$d^2F / d\zeta^2 \approx - s d^2 s / d\zeta^2 \tag{1}$$

is valid along the free boundary, where s is the arc length measured from the vertex of the free boundary ζ_A [5]. Since

$$d\zeta / ds = e^{ig(s)} \tag{2}$$

where g(s) is a real function, then, by virtue of (2.8), (1), and (2)

$$h(\zeta) = \int_0^s \sqrt{isg'(s)}\, ds \tag{3}$$

It is assumed that the free boundary has a curvature of constant sign, namely $g'(s) > 0$. (For $g'(s) < 0$ the asymptotic (2.7) cannot be satisfied.) Then, $\arg h(\zeta) = \pi/4$ on the free boundary by virtue of (3). Furthermore, along the axis of symmetry $\xi = 0$

$$\frac{d^2F}{d\zeta^2} = - \frac{\partial^2 \Phi}{\partial \eta^2}\bigg|_{\xi=0}$$

Under the natural assumption that the velocity on the axis of symmetry is a monotone function of η, the sign of $d^2F/d\xi^2$ agrees with the sign obtained from the asymptotic (2.7)

$$d^2F / d\zeta^2 \approx - (2c_0)^{2-\alpha} \cos \alpha\gamma (-\eta)^{\alpha-2} < 0 \quad \text{for} \quad \eta \to -\infty \tag{4}$$

Hence, along the axis of symmetry

$$h(\zeta) = i \int_{\eta_A}^{\eta} \sqrt{d^2F / d\zeta^2}\, d\eta > 0 \tag{5}$$

and $h(\xi)$ coincides with the real positive axis ($h_1 \geq 0$, $h_2 = 0$). Since $\frac{1}{2} < \alpha < 1$, then it follows from (2.7) that for $\xi \to \infty$ the integral defining $h(\xi)$ diverges and $h(\xi) \to \infty$ as $\xi \to \infty$. It follows from the same asymptotic (2.7) that the image of the flow in the h plane will be the interior of the angle obtained.

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